MATH 2028 Honours Advanced Calculus II
2024-25 Term 1
Problem Set 2

due on Oct 4, 2024 (Friday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through CUHK Blackboard on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Problems to hand in

1. Prove that if $A \subset \mathbb{R}^n$ is compact [1] and has measure zero, then A has content zero.

2. Define the *volume* of a subset $\Omega \subset \mathbb{R}^n$ by $\text{Vol}(\Omega) = \int_\Omega 1\, dV$.

   (a) Let $A \subset \mathbb{R}^n$ be a content zero subset. Prove that A must be bounded. Moreover, show that ∂A has measure zero and $\text{Vol}(A) = 0$.

   (b) Let $B \subset \mathbb{R}^n$ be a bounded subset of measure zero. Suppose ∂B has measure zero. Prove that $\text{Vol}(B) = 0$.

3. Evaluate the following integrals:

   (a) $\int_R \frac{x}{x^2+y}\, dV$ where $R = [0,1] \times [1,3]$

   (b) $\int_0^1 \int_{x^2}^x \frac{x}{1+y^2}\, dydx$

   (c) $\int_0^1 \int_{\sqrt{y}}^1 e^{y/x}\, dxdy$

4. Let $\Omega \subset \mathbb{R}^3$ be the portion of the cube $[0,1] \times [0,1] \times [0,1]$ lying above the plane $y + z = 1$ and below the plane $x + y + z = 2$. Evaluate the integral $\int_\Omega x\, dV$.

5. Let $f : R = [0,1] \times [0,1] \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} 1 & \text{if } y \in \mathbb{Q}, \\ 2x & \text{if } y \notin \mathbb{Q}. \end{cases}$$

   (a) Prove that f is NOT integrable on R.

   (b) Show that each iterated integral $\int_0^1 \int_0^1 f(x,y)\, dxdy$ and $\int_0^1 \overline{\int_0^1} f(x,y)\, dydx$ exist and compute their values.

Suggested Exercises

1. (a) Show that the subset $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ has measure zero.

   (b) Show that $\mathbb{Q}^c \cap [0,1]$ does not have measure zero in $\mathbb{R}$.

[1] A subset A is compact if any open cover of A has a finite subcover. The Heine-Borel Theorem says that a subset in $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

2. Let $f : \Omega \to \mathbb{R}$ be a bounded continuous function defined on a bounded subset $\Omega \subset \mathbb{R}^n$ whose boundary $\partial\Omega$ has measure zero. Suppose Ω is path-connected, i.e. for any $p, q \in \Omega$, there exists a continuous path $\gamma(t) : [0, 1] \to \Omega$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Prove that there exists some $x_0 \in \Omega$ such that
$$\int_\Omega f\ dV = f(x_0)\mathrm{Vol}(\Omega).$$

3. Find the volume of the region in $\mathbb{R}^3$ bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

4. Find the volume of the region in $\mathbb{R}^3$ bounded below by the xy-plane, above by $z = y$, and on the sides by $y = 4 - x^2$.

5. Let $f : \Omega \to \mathbb{R}$ be a C^2 function [2] on an open subset $\Omega \subset \mathbb{R}^2$. Use Fubini's Theorem to prove that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ everywhere in Ω.

6. Let $f : R = [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function. Define another function $F : R \to \mathbb{R}$ such that
$$F(x, y) := \int_{[a,x]\times[c,y]} f\ dV.$$
Compute $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ in the interior of R.

7. Let $f : R = [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function such that $\frac{\partial f}{\partial y}$ is continuous on R. Define $G : [c, d] \to \mathbb{R}$ such that
$$G(y) := \int_a^b f(x, y)\ dx.$$
   (a) Show that G is continuous on $[c, d]$.
   (b) Prove that G is differentiable on (c, d) and $G'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y)\ dx$.

Challenging Exercises

1. The following exercise establishes the theorem that a bounded function $f : R \to \mathbb{R}$ is integrable if and only if f is continuous on R except on a set of measure zero. Let $f : R \to \mathbb{R}$ be a bounded function. For each $p \in R$ and $\delta > 0$, we define the *oscillation of f at p* as
$$o(f, p) = \lim_{\delta\to 0^+}\left(\sup_{x\in B_\delta(p)\cap R} f(x) - \inf_{x\in B_\delta(p)\cap R} f(x)\right).$$
   (a) Show that $o(f, p)$ is well-defined and non-negative. Prove that f is continuous at p if and only if $o(f, p) = 0$.
   (b) For any $\epsilon > 0$, let $D_\epsilon := \{p \in R : o(f, p) \geq \epsilon\}$. Show that D_ϵ is a closed subset and the set of discontinuities D of f is given as $D = \cup_{n=1}^\infty D_{1/n}$.
   (c) Suppose f is integrable on R. Prove that $D_{1/n}$ has content zero for any $n \in \mathbb{N}$. Hence, show that D has measure zero.
   (d) Suppose D has measure zero, prove that f is integrable on R.

[2] Recall that a function f is C^k if all the partial derivatives up to order k exist and are continuous.